Infinite Determinant Methods for Stability Analysis of Periodic-Coefficient Differential Equations

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A method is described for the determination of the regions of the parameter space corresponding to stability and instability of the null solution of a restricted system of linear, periodic-coefficient ordinary homogeneous differential equations. The method is applicable to equations which represent the motion of a completely damped mechanical system. For such systems, there can be no periodic or almost-periodic solutions within the region of the parameter space corresponding to stable null solutions. Accordingly, by establishing those parameter values for which almost-periodic solutions are possible, one can determine the boundaries of the regions in the parameter space corresponding to stable solutions. This is accomplished by substituting an almost-periodic solution as the product of a specific periodic function and a Fourier series, obtaining by numerical search those parameter values which guarantee the existence of such a solution by the vanishing of a suitable approximation of a determinant of infinite size. This method is applied to a current problem in space vehicle dynamics, and comparison is made with a numerical implementation of Floquet theory.

Introduction

CONSIDER a system of first order, linear, periodic-coefficient equations described by the matrix equations

$$\dot{x} = A(t)x \tag{1}$$

$$A(t + T) = A(t) \tag{2}$$

where x is an $n \times 1$ matrix and $A(t)$ is an $n \times n$ matrix of continuous functions of period T, and where dot denotes differentiation with respect to time t.

A basic consequence of the work of Floquet[1,2] is the conclusion that Eq. (1) has at least one solution of the form

$$x^{(i)}(t) = e^{(\ln\lambda_i)t/T}\, p_i(t) \tag{3}$$

with

$$p_i(t + T) = p_i(t) \tag{4}$$

and with λ_i an eigenvalue of the $n \times n$ nonsingular matrix $X(T)$, where $X(t)$ satisfies the equations

$$\dot{X} = A(t)X \tag{5}$$

$$X(0) = E \tag{6}$$

where E is the $n \times n$ identity matrix. Thus the columns of $X(t)$ are independent solutions of Eq. (1).

Furthermore, if $\lambda_1, \ldots, \lambda_m$ are m distinct eigenvalues of $X(T)$, $1 \leqq m \leqq n$, there exist m independent solutions described by Eq. (3), and if $m = n$ these form a fundamental system of solutions.

Equation (3) may be written in the form

$$x^{(i)}(t) = e^{(\ln|\lambda_i|)t/T}\, e^{i(\arg\lambda_i)t/T} p_i(t) \tag{7}$$

to reveal that if the modulus of the complex number λ_i exceeds unity the solution $x^{(i)}(t)$ is unbounded, and the null solution of Eq. (1) is unstable. It may further be shown[2] that this null solution is asymptotically stable if all roots $\lambda_1, \ldots, \lambda_m$ of $X(T)$ have moduli less than unity, and the solution is stable if $|\lambda_i| \leq 1$, $i = 1, \ldots m$, and for any λ_i with $|\lambda_i| = 1$ the multiplicity of the root equals the nullity of the matrix $X(T) - \lambda_i E$.

Application of the noted stability criteria to problems of practical interest usually involves the determination of the matrix $X(T)$ by numerical (digital computer) integration of Eq. (5) from the initial value $X(0) = E$. This integration is then followed by a numerical (digital computer) evaluation of the eigenvalues of $X(T)$, and an assessment of stability or instability. Computer programs for the numerical implementation of Floquet theory have been widely used in recent years for satellite attitude stability studies, beginning with the work of Kane in 1962 (see Ref. 3 for the first published application). A recent example of special interest here is the Floquet study by Mingori[4] of the attitude stability of dual-spin satellites with distributed internal damping.

In practical problems one is normally concerned not with the stability or instability of a given motion of a specific idealized physical system, but with the stability of a range of motions of a class of idealized physical systems described by a range of values of descriptive parameters; in other words one is interested in finding those domains of the parameter space for which the null solution of Eq. (1) is stable. [For convenience, the terms "stable regions" and "unstable regions" are henceforth applied to those regions of the parameter space for which the null solution of Eq. (1) is stable and unstable, respectively.] To determine the stable and unstable regions with a numerical implementation of Floquet theory one must establish a gridwork in the parameter space and separately assess stability for each of the nodal points of the gridwork. Such a procedure is expensive and time consuming, but alternative procedures of comparable rigor and generality have not been forthcoming.

Within the framework of a restricted class of differential equations, Bolotin[5] and others have developed alternative procedures which rely upon knowledge of the form of the solution on the boundary lines in the parameter space delineating stable and unstable regions. A general statement

Received July 28, 1969; revision received October 6, 1969.

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concerning the form of the solution on such boundaries can be made after examination of Eq. (7). Since there must always be an eigenvalue with modulus exceeding unity for instability, and there can be no such eigenvalue for stability, it follows from the assumed continuous dependence of stability on parameter values that there must exist an eigenvalue of modulus unity for parameter values on any boundary between stable and unstable regions. Thus on these boundaries there must exist an almost periodic solution of the form

$$x^{(i)}(t) = e^{i(\arg\lambda_i)t/T}\, p_i(t) \tag{8}$$

In general, of course, such solutions may also exist inside stable and unstable regions.

Bolotin[5] provides extensive studies of problems of elastic stability of structures under parametric excitation; such problems reduce mathematically to the determination of the stability of the null solution of Eq. (1), with certain additional restrictions on $A(t)$. He develops for this purpose a classical method for finding the boundaries of regions of stability and instability in the parameter space by finding those parameter combinations that admit the existence of a periodic solution of Eq. (1). Since this is a special case of the general form of the boundary line solution shown in Eq. (8), this method is not as general as Floquet Analysis. Bolotin's method provides for a restricted class of problems an alternative to Floquet analysis. The purpose of the present paper is to develop an extension of the method of Bolotin, and then to apply the new method to the satellite dynamics problem previously subjected to Floquet analysis by Mingori.[4] Proper development of this extension requires first a brief outline of the "infinite determinant method" as applied by Bolotin.[5]

Infinite Determinant Method for Uncoupled Canonical Systems

A differential equation system is termed canonical if it can be written in the form of Hamilton's canonical equations

$$\dot{q}_i = \partial H/\partial p_i, \quad \dot{p}_i = -\partial H/\partial q_i \tag{9}$$

Such a representation is always possible for mechanical systems under holonomic constraints and conservative forces,[6] in which case the Hamiltonian H is $H(q_1, \ldots, q_{n/2}, p_1, \ldots, p_{n/2}, t)$. When H is taken as the quadratic form in $p_i, q_i, i = 1, \ldots n/2$, and when $H(t + T) = H(t)$, Eq. (9) has the structure of Eq. (1), with condition (2). In application to holonomic, conservative mechanical systems, one may of course find equations of motion in the form of Eq. (1) by methods other than that of Hamilton; such equations are still termed canonical.

For canonical systems, Liapunov's reciprocal root theorem[7,8] can be invoked. This theorem indicates that a canonical system with characteristic root λ_i of multiplicity μ_j must also have characteristic root $1/\lambda_i$ of multiplicity μ_j.

The reciprocal root theorem thus guarantees that the presence of a real root $\lambda_i \neq \pm 1$ implies instability for canonical systems, since either that root or its reciprocal must have modulus exceeding unity. On the boundary marking the transition to stability from instability due to the presence of a real root λ_i, the value of λ_i and its reciprocal must each have the value $+1$, or the value -1, giving a root of multiplicity two on the boundary. Figure 1 illustrates the locus of points in the complex plane occupied by the real numbers λ_i and $1/\lambda_i$ as transition is made in the parameter space from the indicated unstable region to a stability boundary.

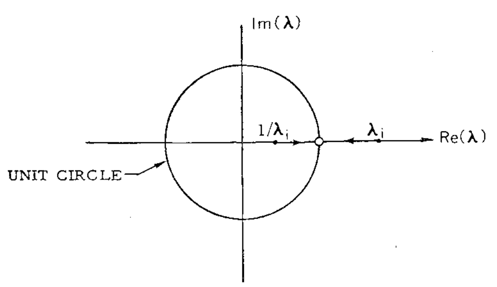

Fig. 1 Locus of real characteristic roots in transition from unstable region to stability boundary.

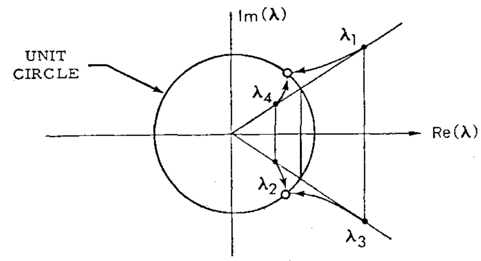

Fig. 2 Locus of complex characteristic roots in transition from unstable region to stability boundary.

It is also possible in general to obtain instability in the absence of real roots. If, however, a complex root (say λ_1) has modulus exceeding unity, there must exist among the roots, in addition to the reciprocal of λ_1 (say λ_2), the complex conjugates of both λ_1 and λ_2. These conjugates (called λ_3 and λ_4 respectively) are required by the presence of only real coefficients in the characteristic polynomial. As is illustrated in Fig. 2, the transition to a stability boundary from a region made unstable by a complex root of modulus exceeding unity is marked by the presence of two roots of multiplicity two.

When one can be assured a priori that for a given problem all stability boundaries are marked by no more than one multiple root, it follows that on all stability boundaries there exist a pair of roots of value $+1$ or -1. If $\lambda_i = +1$ is substituted into the almost periodic function representing the general stability boundary solution in Eq. (8), the result is a periodic function of period T,

$$x^{(i)} = p_i(t) \tag{10}$$

If on the other hand λ_i is -1 on the boundary, Eq. (8) becomes

$$x^{(i)} = e^{i\pi t/T} p_i(t) \tag{11}$$

which is a function of period $2T$. Thus for canonical systems with all stability boundaries marked by no more than one multiple root, these boundaries are characterized by the existence of solutions of period T or $2T$. (Note that not all periodic functions of period $2T$ may be cast in the form of Eq. (11), e.g., a constant does not qualify, so this is a restricted class of functions of period $2T$.)

Two questions raised by the preceding paragraph are the following: 1) Under what conditions is there a priori knowledge that only one pair of multiple roots marks a stability boundary for a canonical system? and 2) under what conditions is there assurance that solutions of the indicated periods will exist only on the boundaries, and not be found also in the middle of stable and unstable regions? If these conditions can be established, the path is clear to a search for stability boundaries in the form of a search for solutions of known period, and established numerical procedures can be applied.

Both of the preceding questions are answered satisfactorily when Eq. (1) is of dimension $n = 2$, since then there are only two characteristic roots in contention and instability because of complex roots (as in Fig. 2) is impossible. A scalar second order differential equation with periodic coefficients (Hill's equation) is in this category, and this may correspond to the equations of motion of a canonical mechanical system with a single degree of freedom. The answer to question 2 is compromised unless one admits explicitly the possibility of a vanishingly small region of stability (instability) within an unstable (stable) region, but this possibility has little practical importance.

The two critical questions are also resolved for uncoupled canonical systems of dimension $n = 2N$, with N an integer representing the number of degrees of freedom of the dynamical system. In other words, if Eq. (1) can be written as the second-order matrix equation

$$I(t)\ddot{q} + K(t)q = 0 \tag{12}$$

with the $N \times N$ matrices $I(t)$ and $K(t)$ diagonal, then the system is no more than a collection of independent second order scalar equations, and the arguments of the preceding paragraph apply to each of the scalar equations individually. The several unstable regions can then be superimposed without ambiguity, even though it may happen that a stability boundary from one scalar equation crosses into a region which is unstable by virtue of another scalar equation.

If however the second-order equations of motion of a canonical system have the structure

$$I(t)\ddot{q} + G(t)\dot{q} + K(t)q = 0 \tag{13}$$

where $G(t)$ is skew-symmetric and $I(t)$ and $K(t)$ are symmetric, there is no assurance that all stability boundaries will be characterized by the existence of solutions of period T or $2T$. Equations of the structure of Eq. (13) are uncommon in the elastic stability problems to which Bolotin has successfully applied a search procedure to identify stability boundaries by the existence of certain periodic solutions. Such equations are however commonplace in space vehicle dynamics problems, where the matrix $G(t)$ originates in the rotation of a body or a reference frame. In the only attempt at the application of Bolotin's procedures to a satellite dynamics problem,[9] the limitations of this method are apparently not fully recognized. The authors state that "the regions of instability are bounded by periodic solutions of period $2T$ and T . . . , " and when the method fails to find all of the known instability boundaries for their problem they simply say that "in its present form this method does not define the regions of [a certain] type . . . " Realizing that this method was somehow inadequate, Meirovitch and Wallace developed other methods to solve their problem, and produced an interesting and valuable paper[9] despite the unexplained partial failure of Bolotin's approach. Clearly more work is needed to permit the identification of those differential equations (1) which admit transformation to Eq. (12) or otherwise meet the requirement of having solutions of period T or $2T$ on all stability boundaries and nowhere else in the parameter space.

The numerical procedure employed[5,9] to find the periodic-solution lines in the parameter space is outlined briefly as follows. If there exists a solution of period T, it must be representable in Fourier series form as

$$x = b_0 + \sum_{k=2,4,6}^{\infty} \left(a_k \sin\frac{k\pi t}{T} + b_k \cos\frac{k\pi t}{T} \right) \tag{14}$$

where b_0, a_k, and b_k are real, constant n $\times$ 1 matrices, of value as yet undetermined. Similarly, any solution of period $2T$ of the character defined by Eq. (11) may be represented as

$$x = \sum_{k=1,3,5}^{\infty} \left(a_k \sin\frac{k\pi t}{T} + b_k \cos\frac{k\pi t}{T} \right) \tag{15}$$

for some values of the constant matrices a_k and b_k.

In principle, one may substitute in turn Eq. (14) and Eq. (15) into Eq. (1), and in each case, by equating to zero the total coefficient of linearly independent trigonometric functions obtain a set of linear, homogeneous algebraic equations for the unknowns a_k and b_k. For arbitrary values of the system parameters these equations have no nontrivial solution, and the determinant of the coefficients of these algebraic equations is nonzero. For those special values of the parameters which admit the assumed periodic solution, the homogeneous algebraic equations do have nontrivial solutions, and the coefficient determinant is zero. The computational task is then a search of the parameter space for parameter values that make a certain determinant zero. The computational obstacle lies of course in the infinite dimension of this determinant, arising from the infinity of terms in the Fourier series. By truncating the Fourier series (and the determinant) quite severely, one may find an approximate indication of the location of lines in the parameter space which correspond to zeros of the infinite determinant. If then the truncation point of the Fourier series is extended and the zeros of the resulting determinants of increasing dimension converge toward some limit set, the infinite determinant evaluation procedure is said to be convergent. Although in some cases convergence can be established formally using the concept of normal determinants,[5] in many applications convergence is established only by successive numerical evaluations.

Infinite Determinant Method for Completely Damped Mechanical Systems

As noted in the preceding section, Bolotin's procedure for finding stability boundaries by searching for solutions of period T and $2T$ is effective in application to uncoupled canonical equations, such as Eq. (12), but the method may fail in application to canonical equations coupled by a skew-symmetric velocity-coefficient matrix, $G(t)$, as in Eq. (13). There are sufficient satellite attitude stability problems of the character of Eq. (13) to warrant an effort to extend Bolotin's version of the infinite determinant method. In fact, if the restriction to canonical systems can be abandoned, one would like to find an infinite determinant method applicable to the matrix equation

$$I(t)\ddot{q} + G(t)\dot{q} + D(t)\dot{q} + K(t)q = 0 \tag{16}$$

where $I(t)$ and $D(t)$, are symmetric, $G(t)$ is skew-symmetric, and all of these matrices are periodic with period T.

Equation (16) corresponds to the equations of motion of a "damped mechanical system." The term "mechanical system" is intended here to describe a system of equations obtained by applying Newton's Second Law or its equivalent to a mathematical model of a physical system. A "damped" mechanical system is a mechanical system which includes m dissipative forces $\mathbf{F}_j$ for which

$$\sum_{j=1}^{m} \mathbf{F}_j \cdot \mathbf{V}_j < 0$$

with $\mathbf{V}_j$ the inertial velocity of the particle to which $\mathbf{F}_j$ is applied at any point in time. Such a system is illustrated for example by the dual-spin satellite equations analyzed by Mingori.[4] Note that the phrase damped mechanical system need not imply a system of second-order linear equations such as Eq. (16). If Hamilton's equations for nonconservative generalized forces or Euler's equations of rotation are applied instead of Newton's Second Law or Lagrange's equations, the resulting equations of motion are first order equations. In any event the equations are generally nonlinear. Equation (16) is illustrative of a class of linearized second-order equations of motion of damped mechanical systems, but more generally such linearized equations may be cast in the form of Eq. (1).

It should be acknowledged that Bolotin[5] offers extension of his method to certain damped mechanical systems which permit time-varying transformation to canonical systems. Briefly, this requires that, in Eq. (16), $G(t)$ be zero and $D(t)$ be restricted (see Ref. 5, pp. 230–249 for restrictions). Such requirements are not often met by those space vehicle attitude stability problems of interest; e.g., they are not met by Mingori's equations for the dual-spin satellite.[4]

There is of course for damped mechanical systems no longer the expectation that all stability boundaries will be characterized by periodic solutions, but there remains the observation that almost-periodic solutions of the form of Eq. (8) must exist on such boundaries. A numerical search of the parameter space for lines or regions permitting the existence of such solutions can be devised (modifying Bolotin's search procedure to accommodate the additional search for a numerical value of arg λ_i for each point in the parameter space), but in general the results are made meaningless by the possibility of finding almost periodic solutions in the stable and unstable regions, as well as on the boundaries. For a certain class of damped mechanical systems, however, one is assured that except on stability boundaries all stable motions are asymptotically stable. Such systems are called "completely damped" or "pervasively damped." Actually these terms are defined on the basis of the sign character of a certain work integral,[10] which assures that for completely damped mechanical systems energy is dissipated for all motions in the neighborhood of the null solution of the equations of motion in the form of Eq. (1), except for that motion corresponding to the null solution proper. Because asymptotic attitude stability is a desirable property of the intended rotational motion of a space vehicle, such vehicles are usually designed to have complete damping. Therefore, a stability analysis procedure applicable only to complete damped mechanical systems retains a wide range of utility.

Almost-periodic solutions clearly cannot exist in a region of the parameter space corresponding to an asymptotically stable null solution of Eq. (1). Almost-periodic solutions might conceivably exist in a region corresponding to an unstable null solution, but physical considerations make this unlikely for a completely damped mechanical system, which must have continual dissipation of mechanical energy for all but the null solution. It may therefore reasonably be argued for such systems that a search for almost-periodic solutions is effectively a search for stability boundaries. Limited experience in application bears out this expectation.

The suggested generalization of Bolotin's search procedure has been programmed for digital computer operation.[11] In order to find lines in the parameter space corresponding to almost-periodic solutions of the form of Eq. (8), one substitutes into Eq. (1) the trial solution

$$x(t) = e^{iht}\left\{b_0 + i\beta_0 + \sum_{k=1}^{\infty} [(a_k + i\alpha_k)\sin 2k\pi t/T + (b_k + i\beta_k)\cos 2k\pi t/T]\right\} \quad (17)$$

where the scalar h is an unknown corresponding to $(\arg\lambda_i)/T$ in Eq. (8), and b_0, β_0, a_k, α_k, b_k, and β_k are real $n \times 1$ matrices. Note that a complex Fourier series is required by the complex character of e^{iht} and the real character of $x(t)$. In comparison with Eq. (14), this results in twice as many unknown column matrices for a given level of Fourier series truncation.

The indicated substitution is followed by equating to zero the coefficients of the real and imaginary parts of the linearly independent products of exponential and trigonometric functions, thereby obtaining a set of homogeneous, linear, algebraic equations for the Fourier coefficients b_0, β_0, a_k, α_k, b_k and β_k, $k = 1, \ldots \infty$. These equations admit a nontrivial solution only when the determinant of their coefficients vanishes. In principle, one might numerically search over a range of values of system parameters and the scalar h in order to find parameters for which this infinite determinant vanishes. In practice, of course, one must truncate the Fourier series in Eq. (17) and thereby truncate the determinant. In application, one must demonstrate that convergence is sufficiently rapid to justify a selected truncation.

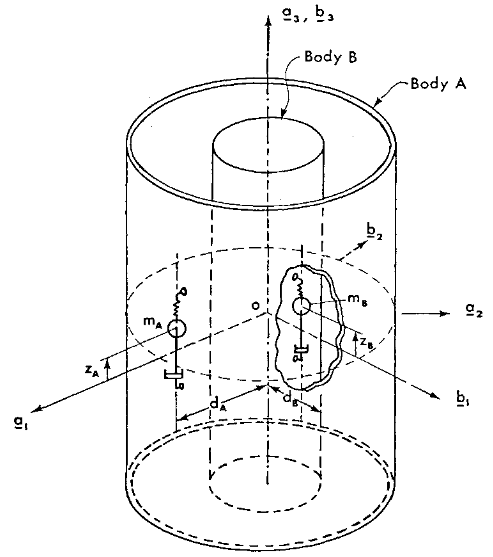

Fig. 3 Description of the dual-spin satellite.

The utility of the proposed infinite determinant method for stability analysis of completely damped mechanical systems depends principally on its computational efficiency. This issue is very difficult to settle in general terms. In order to obtain some indication of the relative merits of this approach and the alternative numerical implementation of Floquet theory, application is made in the next section to a specific problem recently examined in the astronautical literature.[4]

Application: Dual-Spin Satellite with Distributed Damping

Mingori[4] provides a stability analysis of an idealized mechanical system consisting of two rigid bodies constrained to perform relative rotation at a constant rate about a common axis (the bearing axis), while attached to each body there is a mass-spring-dashpot system which permits oscillation of the mass on a line parallel to the bearing axis (see Fig. 3). This system is a conceptual model of the dual-spin satellite, which has assumed great practical significance in the aerospace industry in recent years.

For the special case of symmetric bodies nominally spinning about the bearing axis at different rates, with "tuned" dampers, the nontrivial equations of motion may be written [see Ref. 4, Eqs. (6–9), substituting as indicated in Sec. III].

$$P(t)\dot{x} = Q(t)x \quad (18)$$

with

$$x = [\omega_1{}^*\omega_2{}^*V_A{}^*V_B{}^*Z_A{}^*Z_B{}^*]^T$$

Here $\omega_1{}^*$ and $\omega_2{}^*$ are normalized scalar components of angular velocity transverse to the bearing axis, $Z_A{}^*$ and $Z_B{}^*$ are normalized displacements of the oscillator masses, and $V_A{}^*$ and $V_B{}^*$ are normalized time derivatives of $Z_A{}^*$ and $Z_B{}^*$. The nonzero elements of P and Q are given by

$$P_{11} = 1,\ P_{14} = \delta_B \sin\sigma t,\ P_{22} = 1,\ P_{23} = -\delta_A$$

$$P_{24} = -\delta_B \cos\sigma t,\ P_{32} = -\delta_A,\ P_{33} = \delta_A(1 - \delta_A)$$

$$P_{34} = \delta_A\delta_B,\ P_{41} = \delta_B \sin\sigma t,\ P_{42} = -\delta_B \cos\sigma t,\ P_{43} = \delta_A\delta_B$$

$$P_{44} = \delta_B(1 - \delta_B),\ P_{55} = 1,\ P_{66} = 1$$

$$Q_{12} = 1/(q + Rq') - 1$$

$$Q_{16} = -(\delta_B)[(1 + R)^2/(q + Rq')^2] \sin\sigma t$$

$$Q_{21} = 1 - 1/(q + Rq'),\ Q_{25} = \delta_A/(q + Rq')^2$$

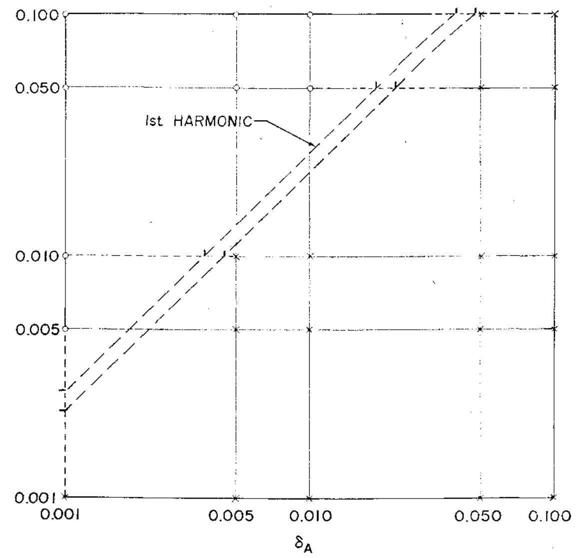

Fig. 4 Stability diagram for the case $R = -6$, $q = 0.8$, $q' = 0.2$.

$$Q_{26} = [\delta_B(1+R)^2/(q+Rq')^2]\cos\sigma t,\ Q_{31} = -\delta_A/(q+Rq')$$

$$Q_{33} = -0.2\delta_A[1 - 1/(q+Rq')](1-\delta_A)^{1/2}$$

$$Q_{35} = -\delta_A(1-\delta_A)(q-1+Rq')^2/(q+Rq')^2$$

$$Q_{41} = -[(\delta_B)(1+2R)/(q+Rq')]\cos\sigma t$$

$$Q_{42} = -[(\delta_B)(1+2R)/(q+Rq')]\sin\sigma t$$

$$Q_{44} = -0.2\delta_B[1-(1+R)/(q+Rq')](1-\delta_B)^{1/2}$$

$$Q_{46} = -\delta_B(1-\delta_B)\{1-(1+R)/(q+Rq')\}^2$$

$$Q_{53} = 1,\ Q_{64} = 1$$

Note that the coefficient matrices $P(t)$ and $Q(t)$ are periodic with period $T = 2\pi/\sigma$, where σ is the rate of prescribed angular rotation between bodies A and B. These matrices involve only the parameters R, q, q', δ_A, and δ_B, for which detailed definitions are available in Refs. 4 and 11 [except that δ_A and δ_B are called δ and δ' in Ref. 4 and R is called (Ω/σ)]. It will suffice for present purposes to note the following: R is a measure of the ratio of the nominal inertial spin rate of body A to the relative spin rate σ; q is the sum of the moments of inertia of bodies A and B about the bearing axis, divided by the transverse inertia of the vehicle; q' is the moment of inertia about the bearing axis of body B alone, divided by the transverse inertia of the vehicle; and δ_A and δ_B are measures of the sizes respectively of the oscillator masses on bodies A and B, in comparison with the size of the total vehicle.

Mingori[4] assesses the stability of the null solution of Eq. (18) for given numerical values of R, q, q', δ_A, and δ_B, using Floquet theory. Results are portrayed in the planar space of parameters δ_A and δ_B (called δ and δ'), thus providing an indication of the influence on stability of damper size. Five stability diagrams in the δ_A, δ_B space are provided in Ref. 4, each for given values of R, q, and q'. Each diagram contains a gridwork of twenty-five points for which stability or instability is actually determined.

In order to test the application of the infinite determinant method to this completely damped mechanical system, all five of the stability diagrams in Ref. 4 were checked with this method.[11] Figures 4–7 illustrate the results for the four cases in which a stability boundary occurs on the diagram (in the fifth case Mingori shows instability everywhere, and indeed nowhere in the given space was the truncated determinant close to zero). In the four figures shown, the large circles on the grid intersections indicate Mingori's determination of stability, while the large X's mark points he found to correspond to unstable motions. The smaller circles and x's on Figs. 6 and 7 are the results of Floquet analysis generated in the course of this study in order to examine a finer gridwork. Each of the four figures contains a dashed-line band marked "first harmonic." These lines delineate regions within which the stability boundary is supposed to lie according to an infinite determinant approximation based on the most severe truncation of the Fourier series in Eq. (17), retaining only the constant and the first harmonic terms. The dashed lines of the gridwork indicate lines along which the search for a zero determinant was actually conducted. In Figs. 4 and 5, the first harmonic approximation is evidently consistent with Mingori's results, but in Fig. 6 there is a minor discrepancy, and in Fig. 7 the discrepancy becomes pronounced for values of δ_B approaching 0.1. For these cases, an improved approximation of the infinite determinant is required. When the second harmonic in the Fourier series is retained, agreement with Mingori is obtained. The shaded bands are the regions within which the stability boundaries should lie according to this approximation. As indicated by the small x's and circles, the accuracy of the infinite determinant method at this level of truncation is extremely high for this problem.

In order to demonstrate the expected inadequacy of the original method of Bolotin in application to this problem, the Fourier series of Eqs. (14) and (15) were used to search for solutions of period T and $2T$. In only one of these five cases was such a solution found, and this one is shown as a solid line in Fig. 7. Note that it follows the stability boundary for $\delta_B < 0.05$, but then it enters the unstable region. This is a surprising result, and one might speculate that improved approximations would indicate that the line of periodic solutions remains on the stability boundary. Indeed, the numerical data indicates that the value of h found in the second harmonic truncation does correspond to a solution of period T.

The figures testify to the accuracy that can be obtained with the modified infinite determinant method advanced in this paper, and illustrate for a specific system the range of accuracy available with the first approximation determinant. There remains the comparative evaluation of digital computer time required by the infinite determinant method and the Floquet method.

It is difficult to offer a definitive comparison, even for a given problem, since so much depends upon the desired accuracy and the computation strategy. For the IBM

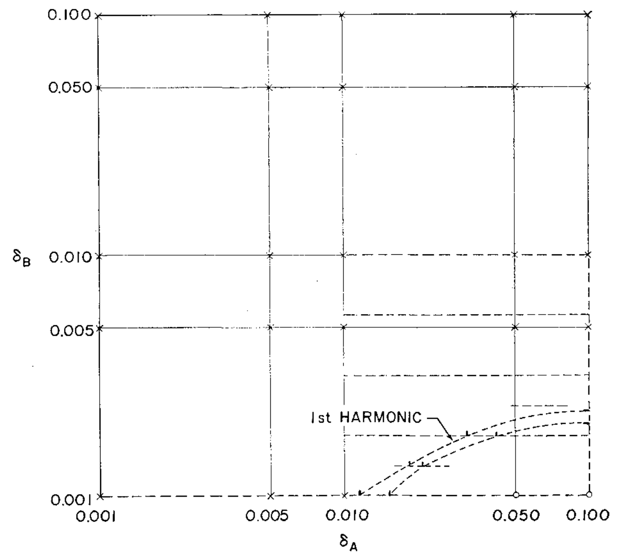

Fig. 5 Stability diagram for the case $R = -2$, $q = 0.8$, $q' = 0.2$.

360/75 programs used in this study, comparisons were made to obtain for a given level of precision in boundary definition an indication of computer time required for the implementation of each method with a "most efficient computational strategy." In order to determine a stability boundary within a bandwidth of $\frac{1}{500}$ of the width of the parameter space used for the preceding figures, 355 sec were required by the Floquet method, whereas 393 sec were required by the infinite determinant method using the second harmonic for the refined location of the boundary. If however, the level of accuracy associated with the first harmonic approximation is acceptable, this figure is reduced to approximately 248 sec. These numbers are measures of the computer time required to locate within a prescribed bandwidth the particular version of a stability boundary which results from each of the three approaches considered. There is no a priori assurance that all three boundaries will be identical; indeed, for this example the Floquet analysis and the infinite determinant second harmonic approximation do give essentially identical results, but the first harmonic approximation of the infinite determinant yields discernibly different results in some portions of some of the figures. If one is interested in preliminary approximations of the domain of stability, so that first harmonic approximations might be suitable, it becomes pointless to locate the approximate boundary to within the high degree of precision defined by $\frac{1}{500}$th of the width of the parameter space. If this number is increased to $\frac{1}{20}$th of the space (which seems from the figures a consistent level of approximation), the Floquet analysis time is reduced from 355 sec to 118 sec, while the first approximation of the infinite determinant finds the boundary in 47 sec, instead of 248 sec. These numbers seem to indicate that for extreme accuracy the Floquet procedure is slightly more efficient than the (second harmonic) infinite determinant method, but for preliminary approximations a substantial computational advantage rests with the infinite determinant method, using only the first harmonic. Aspects of these comparisons not discussed here can be found in Ref. 11.

Conclusions

Certain classes of homogeneous, linear, periodic-coefficient differential equations have solutions of known structure for values of the system parameters which lie on the boundaries in the parameter space separating regions corresponding to stable and unstable null solutions. Two such classes of equations are discussed in this paper.

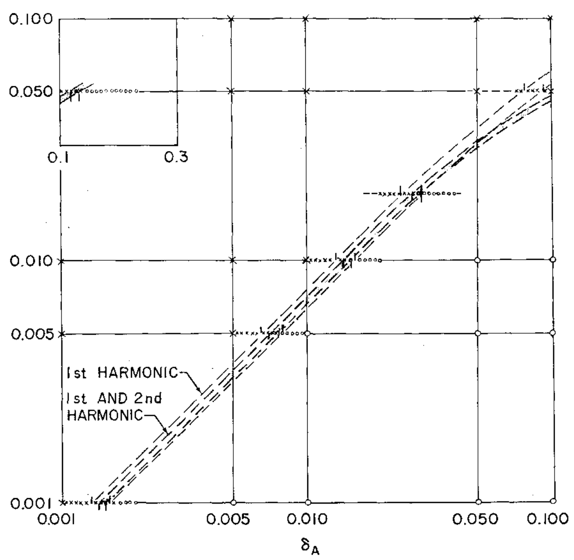

Fig. 6 Stability diagram for the case $R = 0.5$, $q = 0.8$, $q' = 0.2$.

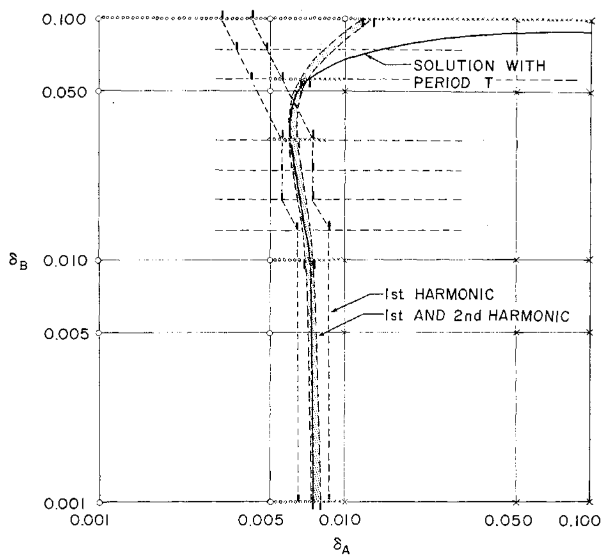

Fig. 7 Stability diagram for the case $R = -1.5$, $q = 1.5$, $q' = 0.2$.

Uncoupled, canonical systems of equations possess solutions of period T or $2T$ on stability boundaries, and not elsewhere [except as noted after Eq. (12)]. Therefore it is possible to discover such boundaries by substituting appropriate Fourier series into the differential equation, seeking those parameters permitting the existence of such solutions. The criterion for this existence is the vanishing of the infinite determinant of coefficients in the algebraic equations arising from the substitution. Bolotin[5] has made frequent successful application of this variant of the infinite determinant method in problems of elastic stability, but application to problems of satellite attitude stability has been thwarted by the character of the equations.

A second version of an infinite determinant method is here proposed for application to completely damped mechanical systems typical of space vehicles. This method involves a search for an almost-periodic solution, which by the nature of the system must exist on all stability boundaries, and cannot exist within regions of stability. This method is successfully applied to an attitude stability problem of current interest, and a comparative analysis is made using a numerical implementation of Floquet theory.

For the single problem examined in detail, computer time required for extremely accurate determination of stability boundaries is about the same for the infinite determinant method and the Floquet procedure (the latter being about ten percent lower in this case). When preliminary results are required, however, and corresponding reductions in required accuracy are acceptable, the proposed infinite determinant method required only 40% of the Floquet analysis time. This suggests that one might in the preliminary design phases of system development use the first harmonic expansion in the infinite determinant method for its computation efficiency, and then resort to the next harmonic for confirmation of results or improvement of the accuracy with which stability boundaries are established. Alternatively, one might adopt the first harmonic infinite determinant approach for system design, and finally use a Floquet analysis to verify the stability of the single design selected.

It must be emphasized that the indicated comparison is based on a single problem of realistic complexity, and this does not permit comprehensive conclusions. It should also be acknowledged that only when the infinite determinant method is clearly more efficient than Floquet analyses can the former method be declared competitive, since in the determination of eigenvalues in the Floquet method one gains more information than simply the existence of a stability boundary. The proposed infinite determinant method accurately locates the lines of transition between stable and unstable regions of the parameter space, but it yields no useful information concerning the character of the solution in regions away from these boundaries.

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Buckling and Postbuckling Behavior of Spherical Caps under Axisymmetric Load

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The elastic buckling and initial postbuckling behavior of clamped shallow spherical shells under axisymmetric load is investigated. An analysis is made of the dependence of the buckling and postbuckling behavior on the area of the region over which the load is uniformly distributed. It is found that as the area of the loaded region increases, the buckling behavior changes from asymmetric bifurcation to axisymmetric snap-through, and then back to asymmetric bifurcation. The asymmetric buckling associated with a small loaded area is characterized by the fact that the shell retains its load carrying capacity. The opposite is true for the bifurcation associated with a relatively large loaded area. A simple criterion for determining whether a loss of load carrying capacity will occur is established in terms of the radius and thickness of the shell and the radius of the projection of the loaded area on the shell base plane.

Introduction

THIS paper presents an analysis of the buckling and initial postbuckling behavior of a clamped spherical cap (Fig. 1) uniformly loaded over a circular region centered at the apex. Huang[1] analyzed the buckling of this structure under uniform pressure. For the case of a concentrated load at the apex, Bushnell[2] analyzed buckling, and Fitch[3] studied the postbuckling as well as the buckling behavior. For both types of loading it was found that if the shell thickness is less than a certain critical value asymmetric bifurcation will occur at a smaller load than that required for axisymmetric snap-buckling. The initial postbuckling analysis for the concentrated load case showed that the shell retains its load carrying capacity as it makes the transition to asymmetric behavior. This result is in qualitative agreement with experiments[4–6] in which the gradual development of asymmetric deformation with increasing load has been observed. Experiments on the clamped cap under uniform pressure,[7–9] on the other hand, have shown that buckling is of the sharp snap-through variety. In this paper an analysis of the dependence

Received January 17, 1969, revision received September 25, 1969. This work was supported in part by NASA under Grant NGL 22-007-012, and by the Division of Engineering and Applied Physics, Harvard University.

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